Fluid Dynamics

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1 Kinematics

1.1 Streamlines and pathlines

There are two natural ways to think of flow.

(i) A stationary observer watching flow go past. This is the Eulerian perspective. This is the approach used through this course. We define a velocity field (continuum field) $\mathbf{u}(\mathbf{x}, t)$.

(ii) A moving observing, travelling along with the flow. This is the Lagrangian perspective.

Definition. (Streamlines) These are curves that are everywhere parallel to the flow at a given instant.

Remark. The streamline that goes through $\mathbf{x}_0$ at time t_0 is given parametrically as $\mathbf{x} = \mathbf{x}(s, \mathbf{x}_0, t_0)$ and

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t_0)$$

(with $\mathbf{x} = \mathbf{x}_0$ at $s = 0$).

The set of streamlines shows the direction of flow a *given* instant a time (all fluid particle at one given time). Take the example $\mathbf{u} = (1, t)$. So at $t = 0$ we have $\mathbf{u} = (1,)$ so the streamlines are horizontal lines. At $t = 1$ we have $\mathbf{u} = (1, 1)$, so the streamlines are diagonal.

Definition. (Pathlines) A *pathline* is the trajectory of a fluid particle (a very small bit of fluid). The pathline $\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0)$ of a fluid which is at $\mathbf{x}_0$ at $t = 0$ is such that

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t)$$

with $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$.

Again if we take $\mathbf{u} = (1, t)$ we get

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = t \end{cases} \rightarrow \begin{cases} x = x_0 + t \\ y = y_0 + \frac{t^2}{2} \end{cases}$$

which describes the path $y - y_0 = \frac{1}{2}(x - x_0)^2$.

Remark. Pathlines are often called "Lagrangian trajectories". The applications are very useful to characterise transport (infecious diseases and pollution simulations).

If the flow is *steady* (so $\mathbf{u}$ does not depend on time). Then pathlines and streamlines are the same.

1.2 The material derivative

We will characterise the rate of change of "stuff" moving with a fluid. Consider a quantity $F(\mathbf{x}, t)$ in a fluid flow (intution is F is temperature). We want to measure how the temperature chnages as we move through the field F along the flow. Let compute the rate of change of (in time) seen by a moving observer. We will call this $\frac{DF}{Dt}$. Take a small time interval δt. Then

$$\begin{aligned}\delta F &= F(\mathbf{x}+\delta\mathbf{x}, t+\delta t) - F(\mathbf{x}, t) \\ &= \delta t\frac{\partial F}{\partial t} + (\delta\mathbf{x}\cdot\nabla)F + (\text{higher order terms}).\end{aligned}$$

We have that $\delta\mathbf{x} = \mathbf{u}\delta t$, so

$$\frac{\delta F}{\delta t} = \frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\mathbf{u}\cdot\nabla)F.$$

We have the derivative and the convected derivative. This should be thought of as moving along gradients of a field.

1.3 Conservation of mass

Consider the flow through a straight rigid pipe with constant cross section. Suppose we have a $\mathbf{u}_{\text{in}}$ and a $\mathbf{u}_{\text{out}}$. Can we have $\mathbf{u}_{\text{in}} \neq \mathbf{u}_{\text{out}}$? For a gas, yes we can since they can be compressed. For a fluid, we cannot, since they are incompressible.

Define $\rho(\mathbf{x}, t)$ as the mass density with $[\rho] = \frac{\mathrm{M}}{\mathrm{L}^3}$. We want a relation between ρ and $\mathbf{u}$. Consider a fixed volume V and compute the rate of change of its mass, M.

$$M = \int_V \rho \mathrm{d}V$$

Assume that mass can only change due to the flow of mass across the boundary surface ∂V. Take a small surface element δA with normal $\mathbf{n}$. The volume out of V during δt is $(\mathbf{u}\cdot\mathbf{n})\delta A\delta t$. Hence the mass out is $\rho(\mathbf{u}\cdot\mathbf{n})\delta A\delta t$, so we get that

$$\frac{dM}{dt} = -\int_{\partial V} \rho(\mathbf{u}\cdot\mathbf{n})\mathrm{d}A.$$

The divergence theorem will allow us to rewrite this as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.$$

We know from IA Vector Calculus that $\nabla\cdot(\rho\mathbf{u}) = \rho\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho$, so we can write that

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}.$$

Definition. (Incompressible) A fluid flow is *incompressible* if $\frac{D\rho}{Dt} = 0$.

This is then equivalent to $\nabla\cdot\mathbf{u} = 0$ which is the equivalent condition we'll use for the course.

For this course we will assume that ρ is constant. This means as a consequence that $\nabla\cdot\mathbf{u} = 0$.

1.4 Kinematic boundary condition

Consider the material boundary, with unit norm $\mathbf{n}$, of a body of fluid has a given velocity $\mathbf{U}(\mathbf{x}, t)$. At a point $\mathbf{x}$ on the boundary, the fluid velocity relative to the surface is $\mathbf{u} - \mathbf{U}$. Applying mass conversation on the interface over a small surface element δA in time δt. So

$$\rho(\mathbf{u} - \mathbf{U})\cdot\mathbf{n}\delta A\delta t = 0.$$

Hence we require $\mathbf{u}\cdot n = \mathbf{U}\cdot\mathbf{n}$ at the interface. This is the kinematic boundary condition.

Remark. $\mathbf{n}$ occurs on both sides, hence we don't need $\mathbf{n}$ to be a unit vector.

We have some consequences of this condition.

(i) If the boundary is fixed, $\mathbf{U} = 0$ implies that $\mathbf{u} \cdot \mathbf{n} = 0$. This is called the no penetration condition.

(ii) Consider an air/water interface (free surface). Suppose the surface is defined by $z = \xi(x, y, t)$. Then can think of the free space as $F(x, y, z, t) = 0$ where $F(x, y, z, t) = z - (x, y, t)$. So $\mathbf{n}$ is perp to $\nabla F = (-\xi_x, -\xi_y, 1)$. Then if $\mathbf{u} = (u, v, w)$ so $\mathbf{U} = (0, 0, \xi_t)$. Then the kinematic boundary condition becomes $-u\xi_x - c\xi_y + w = \xi_t$, so $w = \xi_t + u\xi_x + v\xi_y = \frac{D\xi}{Dt}$. This is equivalent to $\frac{DF}{Dt} = 0$.

1.5 Streamfunction for 2D incompressible flow

We know that $\nabla \cdot \mathbf{u} = 0$ which is equivalent to there existing a vector potential $\mathbf{A}$ such that $\mathbf{u} = \nabla \times \mathbf{A}$. In 2D if $\mathbf{u} = (u, v, 0)$ then $\mathbf{A} = (0, 0, \psi(x, y))$. So

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

We call ψ a *streamfunction*. Looking at dimensions we have that $[\psi] = \mathrm{L}^2\ \mathrm{T}^{-1}$. Now we'll see an example.

Let $\mathbf{u} = (y, x)$ (which we can see is incompressible) so

$$\frac{\partial \psi}{\partial y} = u = y$$

, hence $\psi = \frac{1}{2}y^2 + f(x)$. We also have that $-\frac{\partial \psi}{\partial x} = -f'(x) = x$, so $\psi = \frac{1}{2}(y^2 - x^2) + C$.

We have some properties about the streamfunction,

(i) Streamlines are given by $\psi = \text{constant}$.

(ii) $|\mathbf{u}| = |\nabla \psi|$, so the flow is faster if the streamlines are closer together.

(iii) If we take two points $\mathbf{x}_0, \mathbf{x}_1$, then then the volume flux crossing the line between $\mathbf{x}_0$ and $\mathbf{x}_1$ is

$$\int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u} \cdot \mathbf{n} \mathrm{d}\ell = \psi(\mathbf{x}_1) - \psi(\mathbf{x}_0).$$

(iv) ψ is constant at rigid boundaries.

We can do the same in polar coordinates. So $\mathbf{u} = (u_r(r, \theta), u_\theta(r, \theta), 0)$. We have that

$$\mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0 \right),$$

so we can check that $\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta} = 0$

2 Dynamics of inviscid flow

2.1 Surface and volume forces

There a two types of forces exerted on a fluid.

(i) Forces proportional to the volume (gravity);

(ii) Forces proportional to the surface area (pressure, viscous stresses).

We'll first look at the first type, called volume forces. We'll denote $F(\mathbf{x}, t)\delta V$ as the force acting on a small volume element ∂V. Let's take gravity as an example, so $\mathbf{F} = \rho\mathbf{g}$. Often we have that $\mathbf{F}$ is conservative, so $\mathbf{F} = -\nabla\chi$ for some function χ (we know gravity is $\chi = \rho g z$).

Now for surface forces. Consider a small element of area of $\mathbf{n}\delta A$. Denote the surface force exerted by the positive side on the negative side by $\boldsymbol{\tau}(\mathbf{x}, t, \mathbf{n})\delta A$. We say that $\boldsymbol{\tau}$ is "stress" acting on a surface element. Note that $\boldsymbol{\tau}$ depends on orientation. By Newton's third law, we have that $\boldsymbol{\tau}(\mathbf{x}, t, -\mathbf{n}) = -\boldsymbol{\tau}(\mathbf{x}, t, \mathbf{n})$.

There are many phenomona where friction inside a fluid (viscous stress) is negligible. For example a 10cm box of water, it takes hours for viscosity to bring the fluid to rest once disturbed.

Definition. (Inviscid) A fluid is said to be *inviscid* if we can neglect viscosity.

For inviscid flow, $\boldsymbol{\tau}$ has no tangential component and its magnitude is independent of orientation. So $\boldsymbol{\tau}(\mathbf{x}, t, \mathbf{n}) = -p(\mathbf{x}, t)\mathbf{n}$, where p is the pressure. Note we have the minus sign because the positive side pushes with pressure p towards the negative side when $p > 0$.

2.2 The Euler Momentum equation

The idea here is that we do a similar calculation for mass conservation, but now for momentum instead. Consider an arbitrary fixed volume, V with boundary ∂V. Hence the momentum inside V is

$$\int_V \rho\mathbf{u}\ \mathrm{d}V.$$

The momentum inside V can change due to

(i) Flux of momentum across ∂V;

(ii) Force acting on V or ∂V.

The volume out of δA in δt is $(\mathbf{u}\cdot\mathbf{n})\delta A\ \delta t$. So the momentum out of δA in time δt is $\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\delta A\ \delta t$. Hence we get the following.

Theorem. (Euler momentum integral equation)

$$\frac{d}{dt}\int_V \rho\mathbf{u}\ \mathrm{d}V = -\int_{\partial V} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\ \mathrm{d}A + \underbrace{\int_V \mathbf{f}\mathrm{d}V}_{\text{volume force}} + \underbrace{\int_{\partial V} -p\mathbf{n}\ \mathrm{d}A}_{\text{surface force}}.$$

In components,

$$\int_V \frac{\partial}{\partial t}(\rho u_i)\ \mathrm{d}V = -\int_{\partial V} \rho u_i u_j n_j\ \mathrm{d}A + \int_{\partial V} -p n_i\ \mathrm{d}A + \int_V f_i\ \mathrm{d}V.$$

Sometimes books call $\rho u_i u_j$ the momentum flux tensor. We can apply the divergence theorem to the first two integrals which gives that those two integrals become

$$\int_V \left[-\frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial p}{\partial x_i}\right] \mathrm{d}V.$$

Given that this true for any fixed volume V, the integrand must be zero, hence we have that

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + f_i.$$

This becomes

$$u_i\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right] + \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i\right] = -\frac{\partial p}{\partial x_i} + f_i.$$

So using mass conservation on the first part we get some simplification, so going back to vector form we have the Euler momentum equation,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f}.$$

This is the equation of motion for inviscid fluid flow.

Fluid particles accelerate under differences in pressure and volume (body) forces (inviscid flows only).

Remark. Note that at the surface, the stress exerted by the fluid at the surface is $p\mathbf{n}$.

Let's see an application of the momentum equation. Consider a 90° bent pipe with a flow U. What is the force exerted by flow on the pipe for a steady flow without gravity? We will use the Euler momentum integral equation. Since the flow is steady, the LHS term is zero and since there is no gravity, the volume force term is zero. Hence we have that

$$\int_{\text{walls}} + \int_{\text{end}} [\rho \mathbf{u}(\mathbf{u} \cdot n) p\mathbf{n}]\mathrm{d}A = 0$$

and because of our kinetmatic boundary condition, $\mathbf{u} \cdot \mathbf{n} = 0$, so our integral across the walls becomes

$$\int_{\text{walls}} p\mathbf{n}\ \mathrm{d}A = \mathbf{F} = \text{Force exerted by fluid flow on the pipe.}$$

Across the ends,

$$\int_{\text{in+ out}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p][\mathrm{d}A = \rho(-U)(-U\mathbf{n}_1)A_1 + p_1\mathbf{n}_1 + \rho U(U\mathbf{n}_2)A_2 + p_2\mathbf{n}_2 A_2.$$

W ehave that $p = p_1 = p_2$ and $A_1 = A_2 = A$ so the integral becomes

$$= A[(p + \rho U^2)(\mathbf{n}_1 + \mathbf{n}_2)],$$

hence

$$\mathbf{F} = -A(p + \rho U^2)(\mathbf{n}_1 + \mathbf{n}_2).$$

2.3 Bernoulli equation for steady flow with potential forces

There are *two* Bernoullli equations in the course. We'll look at the first one for steady flow here

Recall from the Euler equation that

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p+\mathbf{F}.$$

We will assume that

(i) We have steady flow so $\frac{\partial}{\partial t}=0$.

(ii) ρ is constant (as always in IB Fluid Dynamics).

(iii) $\mathbf{F}=-\nabla\chi$ (conservative force).

So the Euler equation gives that

$$\rho(\mathbf{u}\cdot\nabla\mathbf{u})=-\nabla(p+\chi).$$

Now we have the identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u}=\nabla\left(\frac{1}{2}u^2\right)-\mathbf{u}\times(\nabla\times\mathbf{u}),$$

which gives that

$$\rho\left[\nabla\left(\frac{1}{2}u^2\right)-\mathbf{u}\times(\nabla\times\mathbf{u})\right]=-\nabla(p+\chi).$$

Now since ρ is constant we can move it inside the ∇. We'll now define $\boldsymbol{\omega}=\nabla\times\mathbf{u}$, the *vorticity* of the fluid, giving,

$$\nabla\left[\frac{1}{2}\rho u^2+p+\chi\right]=\rho\mathbf{u}\times\boldsymbol{\omega}.$$

We like to dot this with $\mathbf{u}$ to get the Steady Bernoulli equation,

$$\mathbf{u}\cdot\nabla\left[\frac{1}{2}\rho u^2+p+\chi\right]=0.$$

So this value $H=\frac{1}{2}\rho u^2+p+\chi$ is constant on streamlines for steady flow. The physical consequence of this is that when u increases, p decreases (ignoring gravity). Let's now see a simple application.

2.3.1 Tank of fluid with small drain

Take a tank full of fluid with a small hole at the bottom. The water height has height h and the fluid exists the tank with speed u. We assume that the size of the hole is small enough so the flow is steady. At the top of the tank, the pressure is atmospherical pressure and the exist it is the same. Taking a streamline from the top on the tank to the exist we'll use the steady flow Bernoulli equation. We have gravity so $\chi=-\rho gh$. At the top we have H_1 and at the bottom we have H_2, from Bernoulli, $H_1=H_2$,

$$\frac{1}{2}\rho u^2=\rho gh,$$

so

$$u=\sqrt{2gh}.$$

Remark. We take the hole very small, so $u=0$ (approximately) at the top of the tank.

2.3.2 Venturi meter

This is a device to measure flow rates with no moving parts. We ignore gravity and assume the flow is steady and uniform across any cross section (gentle variations in the cross section A). The device is a pipe which is pinched in the middle to a much smaller cross section area. Before the pinch we have A_1, u_1, p_1 and in the pinched area we have A_2, u_2, p_2. By conservation of mass, $A_1 u_1 = A_2 u_2$. Now we attach two linked tubes of fluid which have a difference in height of fluid h one at the first point, and the other at the second point. From the Steady Bernoulli equation, $\frac{1}{2}\rho u_1^2 + p_1 = \frac{\rho}{u}{}^2_2 + p_2$. So using our mass conservation,

$$p_1 - p_2 = \frac{1}{2}\rho u_1^2 \left(\frac{A_1^2}{A_2^2} - 1 \right).$$

If we measure h using hydrostatic balance,

$$\rho g h = \frac{1}{2}\rho u_1^2 \left(\frac{A_1^2}{A_2^2} - 1 \right)$$

hence

$$A_1 u_1 = \sqrt{2gh}\frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}}.$$